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Zamolodchikov–Faddeev algebra related to Z_n symmetric elliptic R-matrix

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Abstract. We give representations for the Zamolodchikov–Faddeev algebra (ZFA) in the elliptic case and prove the self-consistency of this algebra in such a case. We also study the implications of our approach for those algebras which are relevant to the extension of the q-deformed affine $A_n^{(1)}$ algebra.

1. Introduction

Zamolodchikov–Faddeev algebras [1–5] (ZFA) are closely related to factorizable scattering problems, Yang–Baxter equations and exactly solvable models, thus they are important for both physics and mathematics.

For a given R-matrix, one may formulate the corresponding relations for a ZFA. The R-matrix satisfies the Yang–Baxter equation (YBE) [1–4] if such a ZFA has a non-degenerate representation. Conversely, if the associated R-matrix satisfies the YBE, we still need to know whether the ZFA has non-trivial representations. The representations of the ZFA are thus crucial for the study. Kulish [5, 6] and Frenkel and Reshetukhin [7] have given the representations of the ZFA for trigonometric R-matrices. The representation of a Zamolodchikov algebra for the elliptic R-matrix associated with an eight-vertex model has also been given in Foda *et al* [8].

We obtain three representations of the ZFA for the elliptic Z_n symmetric R-matrix [9, 10] given by Belavin. The construction of two representations is explained in section 2. In section 3 we study the co-module and a solvable lattice model with non-periodic, non-reflecting boundary conditions and study the fusion of the ZFA, which yields the third representation. In section 4, we construct the L-matrices as generators of the Reshetikhin and Semenov-Tian-Shansky algebra (RSA) [11] and derive some relations similar to the algebraic relations of a quantum affine algebra given by Frenkel and Reshetikhin [7] in the elliptic case. We then study the reflection equation in section 5. Our construction of the representation can also be applied to some algebraic relations of the elliptic quantum algebra given by Foda *et al* [12], which is an extension of the q-deformed affine $A_n^{(1)}$ algebra. This is explained in section 6. (The representation we construct is not a highest-weight representation.)

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2. The Zamolodchikov–Faddeev algebra associated with an elliptic Z_n R-matrix and its representations

2.1. ZFA and R-matrices

The ZFA is the quotient of a tensor algebra generated by $B^i(z)$, $A_i(z)$ modulo the ideal generated by relations (1)–(4). That is, the ZFA is an associative algebra generated by two sets of operators $B^i(z)$, $A_i(z)$ with exchange relations determined by the r-matrices:

$$(r_1)_{ij}^{i'j'}(z_1 - z_2)A_{i'}(z_1)A_{j'}(z_2) = A_j(z_2)A_i(z_1) \tag{1}$$

$$(r_2)_{ij}^{i'j'}(z_1 - z_2)B^i(z_1)B^j(z_2) = B^{j'}(z_2)B^{i'}(z_1) \tag{2}$$

$$(r_3)_{ij}^{i'j'}(z_1 - z_2 - \tfrac{1}{2}nw)B^j(z_2)A_{i'}(z_1) = A_i(z_1)B^{j'}(z_2) \tag{3}$$

$$(r_4)_{ji}^{j'i'}(z_2 - z_1 - \tfrac{1}{2}nw)A_{i'}(z_1)B^j(z_2) = B^{j'}(z_2)A_i(z_1) \tag{4}$$

where $i, j, i', j' = 0, 1, \ldots, n-1$, and summation over repeated indices is assumed. In most of the text, the four r-matrices are the same and defined by Belavin's Z_n elliptic R-matrix [9, 10] and the function $K(z)$ (in section 6, we study the case where the four r-matrices are not identical).

$$(r_m)_{ij}^{i'j'}(z) = r_{ij}^{i'j'}(z) \equiv \frac{R_{ij}^{i'j'}(z)}{K(z)} \qquad m = 1, 2, 3, 4 \tag{5}$$

$$\begin{aligned} R_{ij}^{i'j'}(z) &= \frac{h(z)\theta^{(i'-j')}(z+w)}{\theta^{(i-j')}(z)\theta^{(i'-i)}(w)} \qquad \text{if } i+j = i'+j' \bmod n \\ &= 0 \qquad \text{otherwise} \end{aligned} \tag{6}$$

$$\theta^{(j)}(z) = \sum_{m\in Z} \exp\left\{\pi i n\tau\left(m + \tfrac{1}{2} - \frac{j}{n}\right)^2 + 2\pi i\left(m + \tfrac{1}{2} - \frac{j}{n}\right)(z + \tfrac{1}{2})\right\}$$

$$h(z) = \frac{\prod_{j=0}^{n-1}\theta^{(j)}(z)}{\prod_{j=1}^{n-1}\theta^{(j)}(0)}.$$

The scalar function $K(z)$ satisfying [10, 13–15]:

$$K(z)K(-z) = \frac{h(z+w)h(-z+w)}{h^2(w)} \tag{7a}$$

$$K^{-1}(-z)\prod_{j=1}^{n-1}K(z - jw) = \prod_{j=2}^{n-1}\frac{h(-z+jw)}{h(w)} \tag{7b}$$

$$K(z)K(-z-nw) = \frac{h(-z)h(z+nw)}{h^2(w)} \tag{7c}$$

is given in [10, 15]. The r-matrices satisfy the Yang–Baxter equation (YBE):

$$r_{ij}^{i'j'}(z_1 - z_2)r_{i'k}^{i''k'}(z_1 - z_3)r_{j'k'}^{j''k''}(z_2 - z_3) = r_{jk}^{j'k'}(z_2 - z_3)r_{ik'}^{i'k''}(z_1 - z_3)r_{i'j'}^{i''j''}(z_1 - z_2) \tag{8a}$$

with unitarity and crossing unitarity:

$$r_{ij}^{i'j'}(z)r_{j'i'}^{j''i''}(-z) = \delta_{ii''}\delta_{jj''} \tag{9a}$$

$$r_{ij'}^{i'j''}(z)r_{ji'}^{j'i''}(-z-nw) = \delta_{ii''}\delta_{jj''} \tag{9b}$$

as a result of the properties of $R(z)$ [10] and equation (7).

Equation (9a) is necessary for the self-consistency of (1), (2). Equation (9b) makes (3) and (4) equivalent.

We can depict $r(z)$, $A(z)$, $B(z)$ and (1)–(4) graphically as shown in figure 1. The YBE (8a) is represented by figure 2(a). We also have a crossed YBE:

$$\begin{aligned} &r_{i'j}^{i''j'}(z_1-z_2)r_{k'i}^{k''i'}(z_3-z_1-\tfrac{1}{2}nw)r_{j'k}^{j''k'}(z_2-z_3-\tfrac{1}{2}nw) \\ &\quad = r_{jk'}^{j'k''}(z_2-z_3-\tfrac{1}{2}nw)r_{ki'}^{k'i''}(z_3-z_1-\tfrac{1}{2}nw)r_{ij'}^{i'j''}(z_1-z_2). \end{aligned} \tag{8b}$$

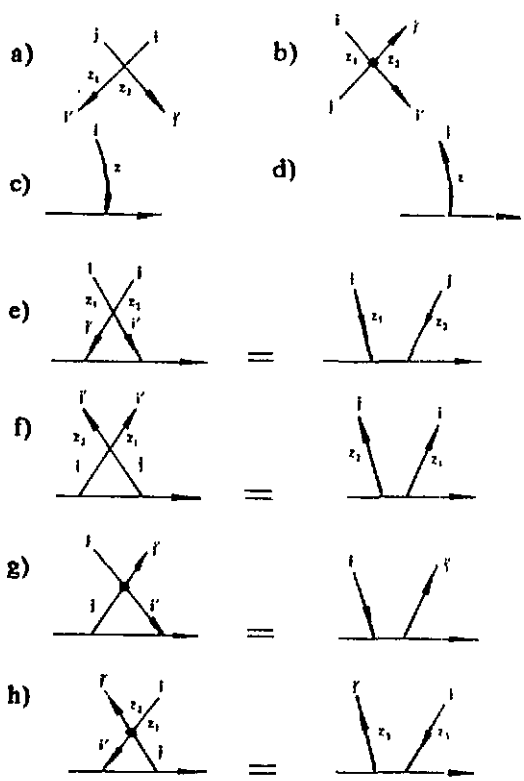

Figure 1. (a) $r_{ij}^{i'j'}(z_1-z_2)$; (b) $r_{ij}^{i'j'}(z_1-z_2-\frac{1}{2}nw)$; (c) $A_i(z)$; (d) $B^j(z)$; (e) equation (1); (f) equation (2); (g) equation (3); (h) equation (4).

This equation can be proved by the crossing unitarity of the elliptical Z_n R-matrix and (8a). Figures 2(b)–(d) are graphic representations of equations (8b) and (9) respectively. The proof of (8b) is given in figure 3.

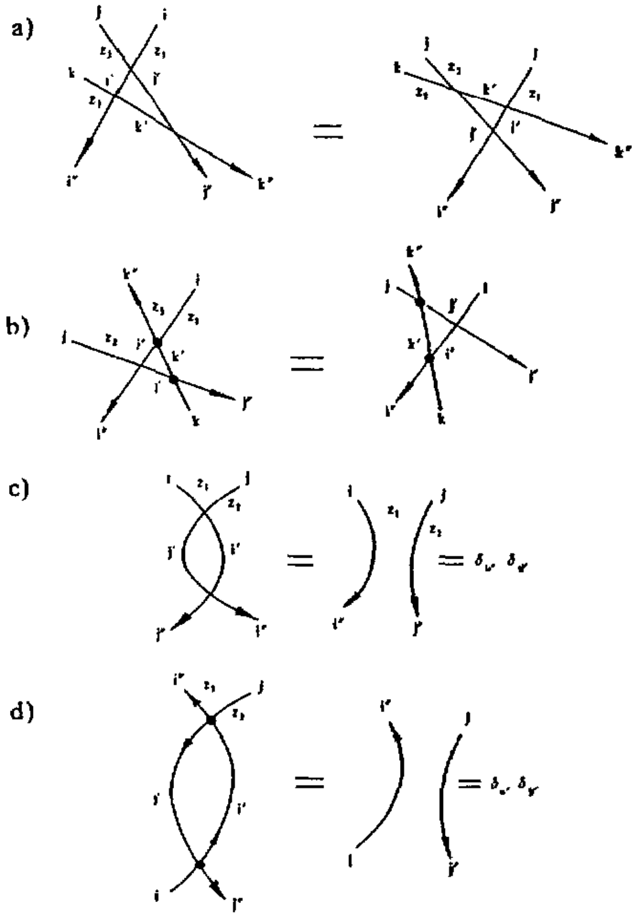

Figure 2 The YBE and the unitarity of the r-matrix: (a) equation (8a); (b) equation (8b); (c) equation (9a); (d) equation (9b).

2.2. Zeros and poles of $K(z)$

$K(z)$ is given in the form of a product expansion in [15]. For simplicity we may further assume that τ, w, z satisfy

$$j\tau + lw + mz = 0 \qquad \Rightarrow j = 0$$

for integers j, l, m. Then the zeros of $K(z)$ are (k is an integer):

$$\begin{aligned} z &= w + knw \qquad k \geqslant 0 \\ z &= -(k+1)nw \qquad k \geqslant 0. \end{aligned} \tag{10a}$$

The poles are

$$\begin{aligned} z &= (k+1)nw \qquad k \geqslant 0 \\ z &= -w-(k+1)nw \qquad k \geqslant 0. \end{aligned} \tag{10b}$$

Thus the zeros of $r(z)$ are at (10b) while the poles are at (10a), since $R(z)$ is always regular and never vanishes. We denote the set of (10a) as $[z_p]$. The points in (10) are singular points of $r(z)$. $r(z)$ always has an inverse and crossing inverse excluding these points.

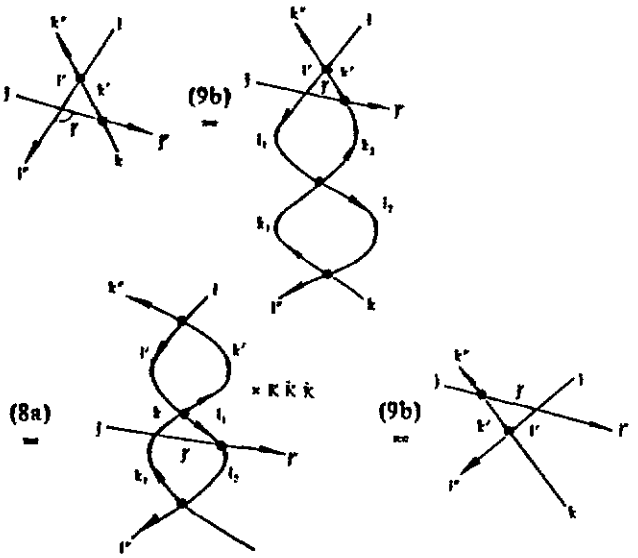

Figure 3. A proof of (8*b*).

2.3. *Normal ordering of monomials of B, A*

For a monomial of B's and A's, we can change the ordering of these operators using (1), (2), (3) and (4). That is, we can expand this monomial by a linear combination of monomials to other orderings, provided the $r(z)$-matrices encountered are not divergent. However, since $r(z)$ has poles, we will not be able to change a given ordering to all other orderings for certain sets of operators. Henceforth we assume that, for a given set of $A(z_a)$'s and $B(z_b)$'s, we can properly arrange the spectra z_a and z_b in order. In this arrangement, spectra of different types of operator (A or B) are regarded as different, even though they are equal in value. For any pair of spectra of the same type of operator, if z precedes z', then

$$z - z' \notin [z_p]. \tag{11a}$$

For any pair of spectra of different types of operator, if z_a (or z_b) precedes z_b (or z_a) then

$$z_b - z_a - \tfrac{1}{2}nw \notin [z_p] \qquad \text{or} \qquad z_a - z_b - \tfrac{1}{2}nw \notin [z_p]. \tag{11b}$$

In (11a,b), $[z_p]$ denotes the set of z's of poles of the r-matrix in (10a). (Note that this arrangement is not always possible. A counter example is: $z_b = z - \frac{1}{2}nw$, $z + \frac{1}{2}nw$, $z_a = z$.)

We then arrange the ZFA generating operators according to the ordering of their spectrum (if z precedes z', then $c(z)$ is to the left of $c(z')$, where c is A and B). The operators are said to be normally ordered if the monomial is arranged in this way. The monomial is, for example, in the form:

$$F \equiv B^{j_{11}}(z_1)B^{j_{12}}(z_1)\dots A_{i_{21}}(z_2)A_{i_{22}}(z_2)\dots A_{i_{l-1,m}}(z_{l-1})B^{j_{l1}}(z_l)A_{i_{l+1,l}}(z_{l+1})\dots A_{i_{N,k}}(z_N) \tag{11c}$$

where z_k precedes $z_{k'}$ in the order described earlier when $k < k'$. (Some A's and B's could be absent for certain z's.)

Proposition 1. Any two adjacent operators can be exchanged to give normal ordering by (1)–(4).

Proof. (i) If they have the same spectrum and the same type, they are already in normal order.

(ii) If they are of the same type but have a different spectrum, we can use (1), (2) to change their order if they are not in normal order. Due to (11a), the r-matrix encountered is not divergent.

(iii) If they are of different type and not in normal order, we can use (3), (4) to change their order. Due to (11c), the r-matrix encountered is again not divergent. □

Thus any monomial can always be expressed as a linear combination of normal ordered monomials.

Remark 1. If we encounter the zero point of the r-matrix in the procedure, we get zero as the result.

2.4. *Fundamental representation of* ZFA

Assume that we have defined the ordering of the spectra, say,

$$(t_1^a) < (t_2^b) < (t_3^b) < (t_4^a) < (t_5^a)\ldots \tag{12a}$$

where (t) denotes the position number of t in the normal ordering. Note that the spectra of different types of operator must be in different positions, even though they are equal in value. For example, in (12a), we may have

$$t_3^b = t_4^a \qquad t_1^a = t_2^b\ldots.$$

We then have normal ordered monomials of the ZFA generators in which all operators are arranged according to the ordering of the spectra. For example,

1 (identity operator of ZFA)

$$\begin{aligned}
&A_{i_1}(t_1^a)A_{i_2}(t_1^a)A_{i_3}(t_1^a)\\
&A_{i_1}(t_2^a)A_{i_2}(t_2^a)\\
&A_{i_1}(t_1^a)A_{i_2}(t_1^a)A_{i_3}(t_4^a)\\
&B^{j_1}(t_2^b)B^{j_2}(t_2^b)A_{i_1}(t_4^a)A_{i_2}(t_5^a)\\
&B^{j_1}(t_2^b)B^{j_2}(t_3^b)A_i(t_5^a)\\
&B^{j_{11}}(z_1)B^{j_{12}}(z_1)\ldots A_{i_{21}}(z_2)A_{i_{22}}(z_2)\ldots A_{i_{l-1,m}}(z_{l-1})B^{j_{l1}}(z_l)A_{i_{l+1,l}}(z_{l+1})\ldots A_{i_{N,k}}(z_N)\\
&\ldots
\end{aligned} \tag{12b}$$

are all normal ordered monomials. Similarly we define vectors as the formal monomials according to the ordering of spectra. For example, corresponding to (12b) we have

$$
\begin{aligned}
&|1\rangle \\
&|a_{i_1}(t_1^a)a_{i_2}(t_1^a)a_{i_3}(t_1^a)\rangle \\
&|a_{i_1}(t_2^a)a_{i_2}(t_2^a)\rangle \\
&|a_{i_1}(t_1^a)a_{i_1}(t_1^a)a_{i_3}(t_4^a)\rangle \qquad\qquad (12c) \\
&|b^{j_1}(t_2^b)b^{j_2}(t_2^b)a_{i_1}(t_4^a)a_{i_2}(t_5^a)\rangle \\
&|b^{j_1}(t_2^b)b^{j_2}(t_3^b)a_i(t_5^a)\rangle \\
&|b^{j_{11}}(z_1)b^{j_{12}}(z_1)\ldots a_{i_{21}}(z_2)a_{i_{22}}(z_2)\ldots a_{i_{l-1,m}}(z_{l-1})b^{j_{l1}}(z_l)a_{i_{l+1,l}}(z_{l+1})\ldots a_{i_{N,k}}(z_N)\rangle .
\end{aligned}
$$

That is each normal ordered monomial of ZFA corresponds to a vector. We regard all these vectors as the base vectors of a space H on which we can construct a representation of ZFA.

Remark 2. Even though we do not know whether the normal ordered monomials of ZFA are linearly independent or not under relations (1)–(4), we still regard all base vectors in (12c) as independent in H.

We next define the action of operators $A_i(t)$ and $B^j(t)$ on the base vectors $|\Psi\rangle$. First, we find the corresponding normal ordered monomial of ZFA which has the same form as Ψ. Then the operators $A_i(t)(B^j(t))$ act on the monomial. We successively exchange the $A(t)(B(t))$ with the operators in the monomial using (1)–(4), until $A(t)(B(t))$ arrives at its correct position. We thus have a unique linear combination of the normal ordered monomials of ZFA, from which we have a corresponding linear combination of base vectors in H. We define this linear combination of base vectors as a resulting vector $A_i(t)|\Psi\rangle$. For example, we have

$$
\begin{aligned}
&A_i(t_1^a)|1\rangle = |a_i(t_1^a)\rangle \\
&A_i(t_2^a)|a_{i_1}(t_2^a)a_{i_2}(t_2^a)\rangle = |a_i(t_2^a)a_{i_1}(t_2^a)a_{i_2}(t_2^a)\rangle \\
&A_i(t_2^a)|a_{i_1}(t_1^a)a_{i_2}(t_1^a)a_{i_3}(t_1^a)\rangle \\
&\qquad = r_{i_1 i}^{i_1' i'}(t_1^a - t_2^a)r_{i_2 i'}^{i_2' i''}(t_1^a - t_2^a)r_{i_3 i''}^{i_3' i'''}(t_1^a - t_2^a)|a_{i_1'}(t_1^a)a_{i_2'}(t_1^a)a_{i_3'}(t_1^a)a_i(t_2^a)\rangle \\
&A_i(t_4^a)|a_{i_1}(t_1^a)a_{i_2}(t_1^a)a_{i_2}(t_4^a)\rangle = r_{i_1 i}^{i_1' i'}(t_1^a - t_4^a)r_{i_2 i'}^{i_2' i''}(t_1^a - t_4^a)|a_{i_1'}(t_1^a)a_{i_2'}(t_1^a)a_{i'}(t_4^a)a_{i_2}(t_4^a)\rangle \\
&A_i(t_5^a)|b^{j_1}(t_2^b)b^{j_2}(t_2^b)a_{i_1}(t_4^a)a_{i_2}(t_5^a)\rangle = r_{i j_1'}^{i' j_1}(t_5^a - t_2^b - \tfrac{1}{2}nw)r_{i' j_2'}^{i'' j_2}(t_5^a - t_2^b - \tfrac{1}{2}nw) \\
&\qquad \times r_{i_1 i''}^{i_1' i'''}(t_4^a - t_5^a)|b^{j_1'}(t_2^b)b^{j_2'}(t_2^b)a_{i'}(t_4^a)a_{i'''}(t_5^a)a_{i_2}(t_5^a)\rangle .
\end{aligned}
$$

The action of $B_j(t)$ is similar:

$$
\begin{aligned}
B^j(t_3^b)|a_{i_1}(t_1^a)a_{i_2}(t_1^a)a_{i_3}(t_4^a)\rangle &= r_{j' i_1}^{j i_1'}(t_3^b - t_1^a - \tfrac{1}{2}nw) \\
&\quad \times r_{j'' i_2}^{j' i_2'}(t_3^b - t_1^a - \tfrac{1}{2}nw)|a_{i_1'}(t_1^a)a_{i_2'}(t_1^a)b^{j''}(t_3^b)a_{i_3}(t_4^a)\rangle .
\end{aligned}
$$

For the vector $|f\rangle$, we have

$$
\begin{aligned}
B^j(z_l)(|f\rangle) &= r_{j_{11}' j^1}^{j_{11} j}(z_1 - z_l)r_{j_{12}' j^2}^{j_{12} j^1}(z_1 - z_l)\ldots r_{j^s i_{21}}^{j^{s-1} i_{21}'}(z_l - z_2 - \tfrac{1}{2}nw) \\
&\quad \ldots r_{j^t i_{l-1,m}}^{j^{t-1} i_{l-1,m}'}(z_l - z_{l-1} - \tfrac{1}{2}nw)|b^{j_{11}'}(z_1)b^{j_{12}'}(z_1)\ldots a_{i_{21}'}(z_2) \\
&\quad \ldots a_{i_{l-1,m}'}(z_{l-1})b^{j^t}(z_l)b^{j_{l1}'}(z_l)\ldots a_{i_N,k}(z_N)\rangle .
\end{aligned}
\tag{13}
$$

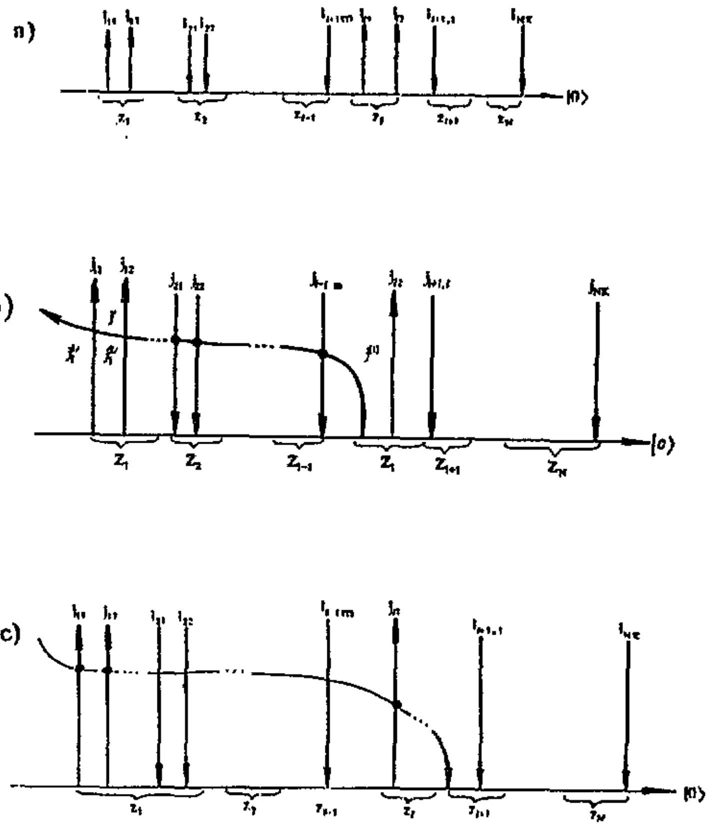

Figure 4. (a) F, a base vector of H; (b) $B^J(z_l)(F)$; and (c) $A_i(z_{l+1})(F)$.

Note that the right-hand side of (13) is a linear combination of normal ordered monomials, which belong to the space H. The vector $A_i(z_l)|f\rangle$ is similarly obtained. They are pictorially expressed in figure 4.

We next check that the action defined above is a representation of ZFA.

The left- and right-hand sides of equations (1)–(4) act on a vector F in H. We assume that the r-matrices in the left-hand side of these equations are well defined. The resulting vector produced by the action of the left-hand side on a vector F is depicted in (bα1, bβ2) of figure 5, which is a linear combination of the basis of H, with the coefficients being *polynomials of the elements of the r-matrices. These r-matrices are depicted as the crossings* of lines in the figure. The vector produced by the action of the right-hand side of equation on the vector F is depicted in (bα3, bβ3) in figure 5.

Note that all r-matrices in the graph are well defined, we can then use (8a), (8b) (which does not yield new r-matrices) and (9a), (9b) (in the proof we need only to eliminate two well defined r-matrices) to prove that (1b) is equivalent to (3b) respectively. In our construction, we need only (8) and (9), which are equivalent to the following statement.

We can reverse the ordering of three operators by two different routes:

$$
\begin{aligned}
&C_1C_2C_3 \to C_2C_1C_3 \to C_2C_3C_1 \to C_3C_2C_1 \\
&C_1C_2C_3 \to C_1C_3C_2 \to C_3C_1C_2 \to C_3C_2C_1
\end{aligned}
\tag{14}
$$

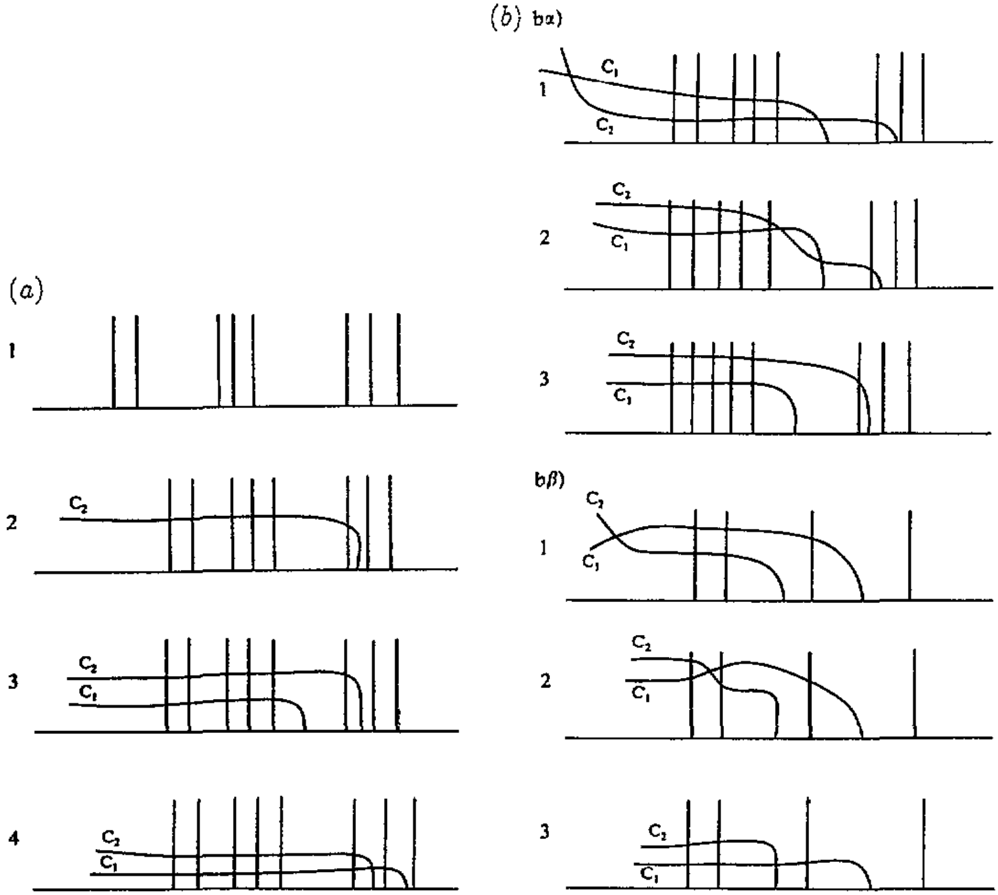

Figure 5. (a) The successive action of two operators: 1, F, a vector of H; 2, C_2F; 3, $C_1(C_2F)$, C_1C_2 is in the right order; 4, $C_1(C_2F)$, C_1C_2 is the wrong order. (b) The action described in figure 6 is a representation of ZFA: $C_1C_2 = r_{12}C_2C_1$: (bα)C_1C_2 is in the right order (normal order); (bβ)C_1C_2 is in the wrong order.

where the C_i's are the generating operators of the ZFA. Thanks to (8*a*) and (8*b*), these two routes give the same final expansion. We also have unitarity and crossing unitarity of $r(z)$ which imply that we can change the order of the two operators twice and recover the original form. We can check whether a ZFA is self-consistent by checking these two relations.

From these we can see that all the relations (1)–(4) are satisfied for the action of the operators in the space H, if the r-matrices in (1)–(4) are well defined, as depicted in figure 5.

Thus we have a representation of the ZFA. We call it the fundamental representation. The ZFA related to the elliptical Z_n symmetric R-matrix is therefore self-consistent.

2.5. PBW base of ZFA

The normal ordered monomials described in section 2.3 together with the identity operator form a PWB base of ZFA. This is because of two facts:

(i) Any monomial can be expressed as a linear combination of the normal ordered monomials.

(ii) The normal ordered monomial acts on the identity vector in H to give a vector, which is a base vector. In addition, different normal ordered monomials yield different base vectors. Thus these normal ordered monomials are linearly independent because the action on H is a representation of ZFA, and the base vectors in H are considered to be linearly independent.

Thus every polynomial of ZFA can be expressed as a linear combination of normal ordered monomials. We thus have these monomials as the PBW base of ZFA.

2.6. *Induced representation*

The two elements of ZFA:

$$e_1(z) \equiv B^i(z - \tfrac{1}{2}nw)A_i(z) \qquad e_2(z) \equiv A_i(z)B^i(z + \tfrac{1}{2}nw) \tag{15}$$

commute with any generating operators of ZFA. We also have from (3), (4):

$$A_i(z)B^j(z - \tfrac{1}{2}nw) = \delta_{ij}e_1(z) \qquad B^i(z + \tfrac{1}{2}nw)A_j(z) = \delta_{ij}e_2(z). \tag{16}$$

However, e_1 and e_2 are not proportional to the identity operator in the fundamental representation, in which all base vectors of (12) are considered to be linearly independent. Next we add two linear relations to these vectors to get an induced vector space H'. These relations are (summation over repeated indices is assumed):

$$|\ldots b^i(z - \tfrac{1}{2}nw)a_i(z)\ldots\rangle = |\ldots 1\ldots\rangle \tag{17a}$$

$$|\ldots a_i(z)b^i(z + \tfrac{1}{2}nw)\ldots\rangle = |\ldots 1\ldots\rangle. \tag{17b}$$

Equation (17) provides many linear relations of the base vectors of the fundamental representation. Figure 6 gives some examples of these relations. For simplicity, in the following we further assume that $z_b = z - \frac{1}{2}nw$ and $z_a = z$ (or $z_a = z$, $z_b = z + \frac{1}{2}nw$) are two successive z's in the normal ordering.

Note that for $z_a = z$, we can have either $z_b = z - \frac{1}{2}nw$ or $z_b = z + \frac{1}{2}nw$ but not both, for otherwise we would have

$$B^i(z + \tfrac{1}{2}nw)A_j(z)B^k(z - \tfrac{1}{2}nw) = \delta_{ij}e_2(z)B^k(z - \tfrac{1}{2}nw) = B^i(z + \tfrac{1}{2}nw)\delta_{jk}e_1(z).$$

If $e_1(z) = e_2(z) = 1$, then $i = j = k$ implies $B^k(z - \frac{1}{2}nw) = 0$, $j = k = i$ implies $B^i(z + \frac{1}{2}nw) = 0$. The statement $z_b = z \pm \frac{1}{2}nw$ is actually still impossible. When $e_1(z)(e_2(z)) = 1$, (15) and (16) imply that any non-trivial representation of such A, B must be infinite dimensional.

We must verify that (17) is consistent with the algebra. For example, let $A_{j_0}(z)$ act on (17a), then

$$\begin{aligned}
\text{LHS} &= A_{j_0}(z)|\ldots b^i(z - \tfrac{1}{2}nw)a_i(z)\ldots\rangle \\
&= \ldots r \ldots r_{jl'}^{j'l}(0)|\ldots b^{i'}(z - \tfrac{1}{2}nw)\ldots a_{j'}(z)a_i(z)\ldots\rangle \\
&= r \ldots \delta_{jl}\delta_{j'i'}|\ldots b^{i'}(z - \tfrac{1}{2}nw)a_{j'}(z)a_i(z)\ldots\rangle \\
&= r \ldots |\ldots b^{j'}(z - \tfrac{1}{2}nw)a_{j'}(z)a_j(z)\ldots\rangle \\
\text{RHS} &= A_{j_0}(z)|\ldots 1\ldots\rangle \\
&= r \ldots |\ldots a_j(z)\ldots\rangle \\
&= \text{LHS}.
\end{aligned}$$

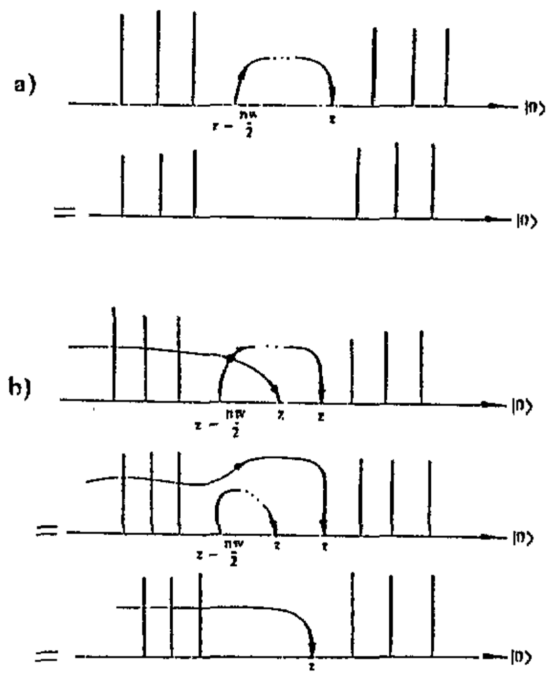

Figure 6. (a) Linear relation (17) of the vectors which makes some vectors in H null. (b) $A(z)$ acting on equivalent vectors still gives equivalent vectors.

One can similarly check other cases. Thus the null vectors of H' still yield null vectors, i.e. equation (17) is consistent with the ZFA.

Next we prove that H' is not totally null. For this purpose we first define $\langle\psi|$ as a dual vector of space H,

$$\langle\psi|f\rangle = (n)^{-m/2}$$

for each base vector corresponding to the normal ordered monomial $|f\rangle$, where m is the number of generators in F. We have

$$\langle\psi|(|\ldots b^i(z - \tfrac{1}{2}nw)a_i(z)\ldots\rangle - |\ldots 1\ldots\rangle) = 0.$$

Thus $\langle\psi|\varphi\rangle = 0$ if $|\psi\rangle$ is null due to (17). However, $\langle\psi|f\rangle = (n)^{-m/2} \neq 0$ shows that there are still many vectors which are not null under the equations (17). Thus H' is not null. In this way, we have established a representation of ZFA where the central element $e_1(z) = 1(e_2(z) = 1)$. After properly renormalizing $B(z)$ and $A(z)$ we can change $e_1(z)(e_2(z))$ to other given functions of z.

3. Co-module and fusion of ZFA

3.1. L-operator and co-module of ZFA

For the matrix operator $L(z)_i^j$ and $S(L(z))_i^j$, acting on H_0, satisfying

$$S(z)_i^j L(z)_j^k = \delta_{ik}\mathrm{id} = L(z)_i^j S(z)_j^k$$

$$r_{ij}^{i'j'}(z_1 - z_2)L(z_1)_{i'}^{i''}L(z_2)_{j'}^{j''} = L(z_2)_j^{j'}L(z_1)_i^{i'}r_{i'j'}^{i''j''}(z_1 - z_2) \qquad (18a)$$

giving

$$S(z_2)_j^{j'} r_{ij'}^{i'j''}(z_1-z_2)L(z_1)_{i'}^{i''} = L(z_1)_i^{i'} r_{i'j}^{i''j'}(z_1-z_2)S(z_2)_{j'}^{j''} \tag{18a}$$

$$L(z_1)_{i'}^{i''} r_{j''i}^{j''i'}(z_2-z_1-nw)S(z_2)_j^{j'} = S(z_2)_{j'}^{j''} r_{ji'}^{j'i''}(z_2-z_1-nw)L(z_1)_i^{i'} \tag{18c}$$

$$S(z_1)_i^{i'} S(z_2)_j^{j'} r_{i'j'}^{i''j''}(z_1-z_2) = r_{ij}^{i'j'}(z_1-z_2)S(z_2)_{j'}^{j''} S(z_1)_{i'}^{i''} \tag{18d}$$

and $A(z)$ and $B(z)$ acting on H', satisfying ZFA, we can construct a co-module of A, B as

$$a_i(z) = L(z)_i^{i'} A_{i'}(z) \qquad b^i(z) = B^{i'}(z)S(z+\tfrac{1}{2}nw)_{i'}^{i}$$

acting on $H_0 \otimes H'$, also satisfying the ZFA. The proof is straightforward. This co-module is a general property of the ZFA. Note that $r(z)$ itself can satisfy equation (18) and thus we can construct a co-module with the $r(z)$ matrix.

3.2. A solvable model

Consider an antiautomorphism operator of $A_i(z)$ and regard it as the transportation of $A_i(z)$ in the Hilbert space (quantum space). We denote such operators as $A^T(z)_i$. We similarly define $B^T(z)^j$. The relation of A^T, B^T can also be depicted by that of A, B, if we determine the direction of the action in the Hilbert space as the opposite direction of the arrow (note that the following derivation of A^T, B^T is valid for any antiautomorphism of A, B). A representation of ZFA suggests an exactly solvable lattice model with a corresponding r-matrix. Suppose that the transfer matrix

$$T \in \mathrm{End}(H'^* \otimes H_0 \otimes H_0 \otimes \ldots \otimes H_0 \otimes H')$$

is formed by

$$T(z) = \sum_{i_1 \ldots i_n} B^T(z)^{i_1} L^{(1)}(z+\delta_1)_{i_1}^{i_2} L^{(2)}(z+\delta_2)_{i_2}^{i_3} \ldots L^{(N)}(z+\delta_N)_{i_{N-1}}^{i_N} A_{i_N}(z+\delta) \tag{19}$$

then by (1)–(4) and (18) we can show that $T(z)$ and $T(z')$ can commute with each other; i.e.

$$[T(z), T(z')] = 0. \tag{20}$$

Thus they may have the same set of complete eigenstates, suggesting a solvable lattice model with a non-periodic, non-reflecting boundary condition.

3.3. Fusion of the generating operators and a third representation

By fusion of the $(n-1)$ r-matrices, we can obtain one r-matrix with the direction of the action on one space reversed. Let $(i'_1, i'_2, \ldots, i'_{n-1}) = P'(0, 1, \ldots, \bar{i}'-1, \bar{i}'+1, \ldots, n-1)$ be a permutation of $(n-1)$ unequal numbers. By fusion of the r-matrices and (7a)–(7c), one can show [14–17] that when $(i_1, i_2, \ldots) = P(0, 1, \ldots, \bar{i}-1, \bar{i}+1, \ldots, n-1) = 0$

$$(-1)^{\bar{i}'} \sum_{p'} (-1)^{p'} r_{i_1 j}^{i'_1 j^{-1}}(z-w) r_{i_2 j_1}^{i'_2 j^2}(z-2w) \ldots r_{i_{n-1} j^{n-2}}^{i'_{n-1} j^{n-1}}(z-(n-1)w) = (-1)^{\bar{i}}(-1)^p r_{j\bar{i}'}^{j^{n-1}\bar{i}}(-z). \tag{21}$$

Otherwise, when

$$i_l = i_m \qquad \text{for some } l, m \in Z_n$$

the left-hand side of (21) equals zero.

Using this equation, we can compose an operator $\bar{A}_{\bar{i}}(z)$ by $B(z)$'s, which have the same exchange properties as the $A(z)$'s. We have

$$\begin{aligned} S \equiv B^{j^{n-1}}(z')\left[(-1)^{\bar{i}'}\sum_{p'}(-1)^{p'}B^{i'_{n-1}}(z-(n-1)w)\ldots B^{i'_1}(z-w)\right] \\ = (-1)^{\bar{i}'}\sum_{p'}(-1)^{p'}\sum_{i_1\ldots i_{n-1}} r_{i_1 j}^{i'_1 j^1}(z-z'-w)r_{i_2 j-1}^{i'_2 j^2}(z-z'-2w) \\ \ldots r_{i_{n-1} j^{n-2}}^{i'_{n-1} j^{n-1}}(z-z'-(n-1)w)B^{i_{n-1}}(z-(n-1)w)\ldots B^{i_1}(z-w)B^j(z') \end{aligned}$$

giving

$$S = \sum_{\bar{i}p}(-1)^{\bar{i}}r_{j\bar{i}'}^{j^{n-1}\bar{i}}(z'-z)(-1)^p B^{i_{n-1}}(z-(n-1)w)\ldots B^{i_1}(z-w)B^j(z')$$

with $(i, \ldots, i_{n-1}) = P(0, 1, \ldots, \bar{i}-1, \bar{i}+1, \ldots)$. Thus

$$S = \sum_{\bar{i}} r_{j\bar{i}'}^{j^{n-1}\bar{i}}(z'-z)\left[(-1)^{\bar{i}}\sum_p(-1)^p B^{i_{n-1}}(z-(n-1)w)\ldots B^{i_1}(z-w)\right]B^j(z').$$

Let

$$(-1)^{\bar{i}}\sum_p(-1)^p B^{i_{n-1}}(z-(n-1)w)\ldots B^{i_1}(z-w) \equiv \bar{A}_{\bar{i}}(z-\tfrac{1}{2}nw). \tag{22}$$

We have

$$B^{j^{n-1}}(z')\bar{A}_{\bar{i}'}(z-\tfrac{1}{2}nw) = \sum_{\bar{i}} r_{j\bar{i}'}^{n^{n-1}\bar{i}}(z'-z)\bar{A}_{\bar{i}}(z-\tfrac{1}{2}nw)B^j(z'). \tag{23}$$

Applying the same procedure to $B(z')$ as that used in (23), we can verify

$$\bar{A}_{\bar{j}}(z')\bar{A}_{\bar{i}}(z) = r_{\bar{i}\bar{j}}^{\bar{i}'\bar{j}'}(z-z')\bar{A}_{\bar{i}'}(z)\bar{A}_{\bar{j}'}(z'). \tag{24}$$

Equations (23) and (24) are the same as (4), (1). Thus, we can fuse $(n-1)$ B's to obtain the $\bar{A}$'s. Similarly, by skew antisymmetrizing $(n-1)$ A's, we can obtain $\bar{B}^j(z)$ with the same exchange properties as the B's. In particular, we have

$$\begin{aligned} \bar{A}_{\bar{i}}(z-\tfrac{1}{2}nw)B^{\bar{i}}(z) = \bar{e}_2(z-\tfrac{1}{2}nw) \\ \bar{B}^{\bar{i}}(z-\tfrac{1}{2}nw)A_{\bar{i}}(z) = \bar{e}_1(z) \end{aligned} \tag{25}$$

which are the central elements of the ZFA. One can also impose them to be constants in certain representations. In this way, we can get a representation of the full ZFA in a representation with only one type of generator B (or A). So far we have constructed the third representation of ZFA.

4. Construction of the RSA

Reshetikhin and Semenov-Tian-Shansky's algebra (RSA) is generated by $L^{\pm}(z)$ and $\tilde{L}^{\pm}(z)$ satisfying [11]

$$
\begin{aligned}
&R(z_1-z_2)L_1^{\pm}(z_1)L_2^{\pm}(z_2)=L_2^{\pm}(z_2)L_1^{\pm}(z_1)R(z_1-z_2)\\
&R(z_1-z_2+\tfrac{1}{2}\Delta)L_1^{+}(z_1)L_2^{-}(z_2)=L_2^{-}(z_2)L_1^{+}(z_1)R(z_1-z_2-\tfrac{1}{2}\Delta)\\
&L_1^{\pm}(z_1)R(z_1-z_2)\tilde{L}_2^{t\pm}(z_2)=\tilde{L}_2^{t\pm}(z_2)R(z_1-z_2)L_1^{\pm}(z_1)\\
&L_1^{+}(z_1)R(z_1-z_2-\tfrac{1}{2}\Delta)\tilde{L}_2^{t-}(z_2)=\tilde{L}_2^{t-}(z_2)R(z_1-z_2+\tfrac{1}{2}\Delta)L_1^{+}(z_1)\\
&\tilde{L}_1^{t+}(z_1)R_{21}(z_2-z_1-\tfrac{1}{2}\Delta)L_2^{-}(z_2)=L_2^{-}(z_2)R_{21}(z_2-z_1+\tfrac{1}{2}\Delta)\tilde{L}_1^{t+}(z_1)\\
&\tilde{L}_1^{t+}(z_1)\tilde{L}_2^{t-}(z_2)R(z_1-z_2+\tfrac{1}{2}\Delta)=R(z_1-z_2-\tfrac{1}{2}\Delta)\tilde{L}_2^{t-}(z_2)\tilde{L}_1^{t+}(z_1)
\end{aligned}
\tag{26}
$$

where

$$
\begin{aligned}
&R(z)\in \mathrm{End}(V\otimes V) \qquad L_1(z)=L(z)\otimes I\\
&L_2(z)=I\otimes L(z) \qquad \text{etc}\\
&R_{21}(z)=PR(z)P \qquad P(e_i\otimes e_j)=(e_j\otimes e_i).
\end{aligned}
$$

Assume $R(z)$ in (26) equals $r(z)$ in (5) and construct L operators by A's and B's acting on $H'\otimes H'$:

$$
\begin{aligned}
L^{+}(z)_i^j&=A_i(z+\tfrac{1}{2}\Delta)\otimes B^T(z+\tfrac{1}{2}nw)^j\\
L^{-}(z)_i^j&=A_i(z)\otimes B^T(z+\tfrac{1}{2}nw+\tfrac{1}{2}\Delta)^j\\
\tilde{L}^{t+}(z)_i^j&=B^j(z-\tfrac{1}{2}nw+\tfrac{1}{2}\Delta)\otimes A^T(z)_i\\
\tilde{L}^{t-}(z)_i^j&=B^j(z-\tfrac{1}{2}nw)\otimes A^T(z+\tfrac{1}{2}\Delta)_i.
\end{aligned}
\tag{27}
$$

Then the L operators satisfy (26).

Furthermore, we also have

$$L^{\pm}(z)_i^j\tilde{L}^{t\pm}(z)_j^k=\delta_{ik}$$

as required by equation (8) in [11]. The difference equation of $A_i(z)$:

$$L^{+}(z+\tfrac{1}{2}\Delta)_i^{i'}(A_j(z)\otimes I)\tilde{L}^{t-}(z)_{i'}^{j}=nA_i(z+\Delta)$$

and the exchange relations of $A_i(z)$ and $\tilde{L}$:

$$
\begin{aligned}
&(A_i(z)\otimes I)L^{+}(z')_j^{j''}=r_{ji}^{j'i'}(z'-z+\tfrac{1}{2}\Delta)L^{+}(z')_{j'}^{j''}(A_i'(z)\otimes I)\\
&\tilde{L}^{t-}(z')_{j''}^{j}(A_i(z)\otimes I)=r_{j'i}^{ji'}(z'-z-nw)(A_{i'}(z)\otimes I)\tilde{L}^{t-}(z')_{j''}^{j'}
\end{aligned}
$$

can be derived from (27) and (1)–(4). These equations are similar to equations (5.1) and (4.47) in [7].

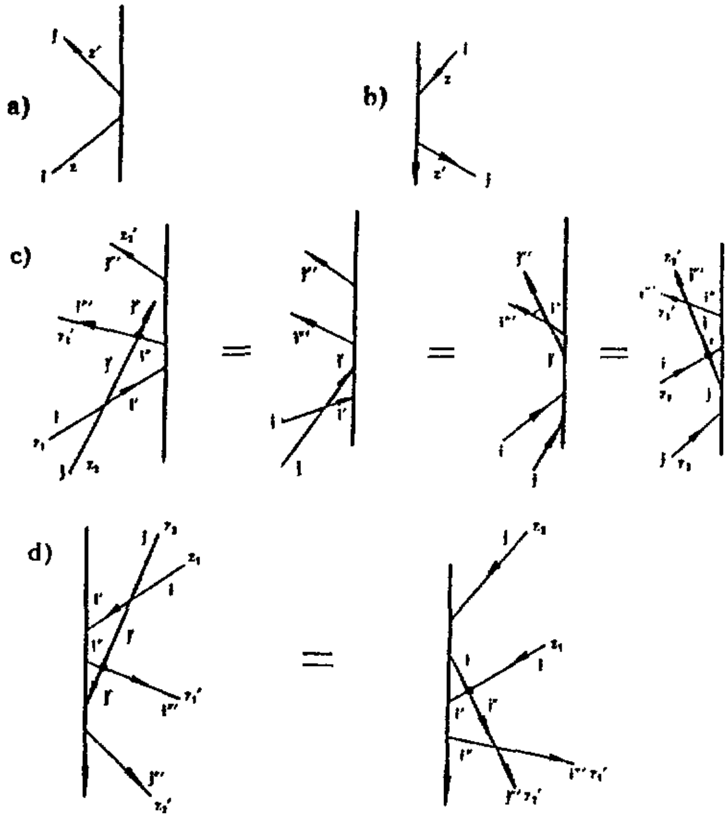

Figure 7. (a) $K(zz')$; (b) $\tilde{K}(zz')$; (c) proof of reflection equation (29*a*); and (d) equation (29*b*).

5. Reflection equation and solvable model

5.1. Reflection equation

From the RSA, we can similarly obtain the operator $L(z)$ that obeys a reflection equation given in [11]:

$$L(z) = L^{+}(z + \tfrac{1}{2}\Delta)\tilde{L}^{t-}(z)$$

$$r_{12}(z_1 - z_2)L_1(z_1)r_{21}(z_2 - z_1 + \Delta + \tfrac{1}{2}nw)L_2(z_2)$$
$$= L_2(z_2)r_{12}(z_1 - z_2 + \Delta + \tfrac{1}{2}nw)L_1(z_1)r_{21}(z_2 - z_1). \tag{28}$$

We can obtain operators that obey the reflection equation in a simpler way. If we let

$$K(zz')_i^j = A_i(z)B^j(z') \qquad \tilde{K}(zz')_i^j = B^T(z')^j A^T(z),$$

we have

$$r_{ij}^{i'j'}(z_1 - z_2)K(z_1z_1')_{i'}^{i''}r_{j'i''}^{j''i'''}(z_2 - z_1' - \tfrac{1}{2}nw)K(z_2z_2')_{j''}^{j'''}$$
$$= K(z_2z_2')_j^{j'}r_{ij'}^{i'j''}(z_1 - z_2' - \tfrac{1}{2}nw)K(z_1z_1')_{i'}^{i''}r_{j''i''}^{j'''i'''}(z_2' - z_1'). \tag{29a}$$

Similarly, $\bar{K}$ satisfies

$$\bar{K}(z_2z_2')^{j'''}_{j''} r^{j''i'''}_{j'i''}(z_2 - z_1' - \tfrac{1}{2}nw)\bar{K}(z_1z_1')^{i''}_{i'} r^{i'j'}_{ij}(z_1 - z_2)$$
$$= r^{j'''i'''}_{j''i''}(z_2' - z_1')\bar{K}(z_1z_1')^{i''}_{i'} r^{i'j''}_{ij'}(z_1 - z_2' - \tfrac{1}{2}nw)\bar{K}(z_2z_2')^{j'}_{j}. \tag{29b}$$

The left-hand side of (29*a*) is

$$r^{i'j'}_{ij}(z_1 - z_2)A_{i'}(z_1)B^{i''}(z_1')r^{j''i'''}_{j'i''}(z_2 - z_1' - \tfrac{1}{2}nw)A_{j''}(z_2)B^{j'''}(z_2')$$
$$\stackrel{(3)}{=} r^{i'j'}_{ij}(z_2 - z_2)A_{i'}(z_1)A_{j'}(z_2)B^{i'''}(z_1')B^{j'''}(z_2')$$
$$\stackrel{(1)(2)}{=} A_j(z_2)A_i(z_1)B^{j''}(z_2')B^{i''}(z_1')r^{j'''i'''}_{j''i''}(z_2' - z_1')$$
$$\stackrel{(3)}{=} A_j(z_2)B^{j'}(z_2')r^{i'j''}_{ij'}(z_1 - z_1' - \tfrac{1}{2}nw)A_{i'}(z_1)B^{i''}(z_1')r^{j'''i'''}_{j''i''}(z_2' - z_1')$$
$$= \text{RHS}.$$

The proof of (29*b*) is similar. This is shown in figure 7.

5.2. *Solvable model with a reflecting boundary condition*

We may use representations of the ZFA as the left-hand and right-hand sides of boundaries to construct a solvable model with reflecting boundary conditions. It is easy to see that the operator

$$r^{i'j_1'}_{ij_1}(z - z_1)r^{i''j_2'}_{i'j_2}(z - z_2)\ldots r^{i^{(N)}j_N'}_{i^{(N-1)}j_N}(z - z_N)A_{i^{(N)}}(z) \equiv a_i(z)^{j_1'\ldots}_{j_1\ldots}$$

is a co-module of $A_i(z)$, acting on $V_1 \otimes \ldots \otimes V_N \otimes H'$. We construct the transfer matrix, acting on $H'^* \otimes V_1 \otimes \ldots \otimes V_N \otimes H'$,

$$T(z) = \bar{K}(z, z + \delta)^j_i \otimes K(z + \delta, z)^i_j$$
$$\bar{K}(z, z')^j_i = B^T(z')^j A^T(z)_i \qquad K(z, z')^i_j = a_i(z)B(z')^j.$$

Then the transfer matrices $T(z)$ and $T(z')$ commute; i.e.

$$T(z')T(z) = T(z)T(z') \tag{30}$$

for arbitrary z, z'. The proof is depicted in figure 8. We thus have a solvable model with a reflecting boundary condition [17–20].

6. Discussion

We may put $B^i(z) = A_{j+n}(z)$ in equations (1)–(4) and rewrite the equations:

$$S^{i'j'}_{ij}(z_1 - z_2)A_{i'}(z_1)A_{j'}(z_2) = A_j(z_2)A_i(z_1) \tag{31}$$

where S is composed by r-matrices. Our construction shows that, provided S satisfies the YBE and the unitarity condition, we can generically have a fundamental representation of

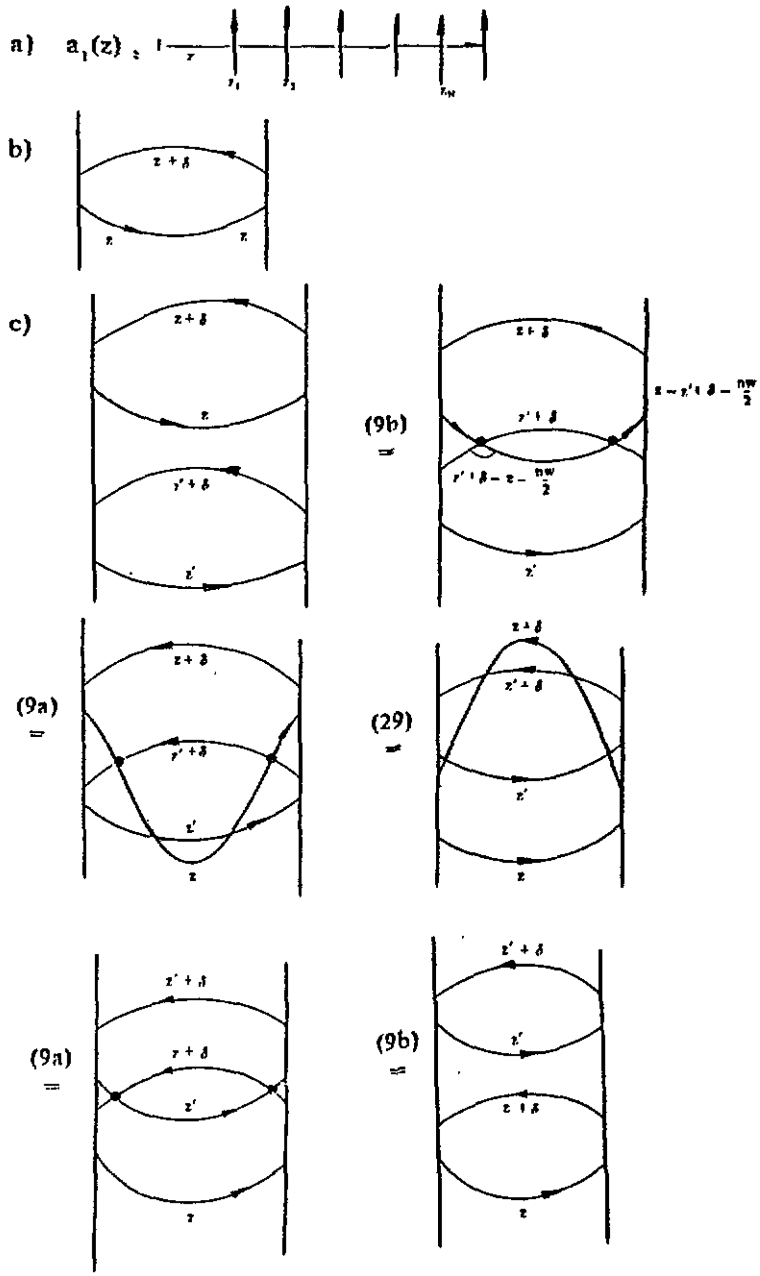

Figure 8. (a) $a_i(z)$, the co-module of $A_i(z)$; (b) $T(z)$; and (c) proof of (30) using (9) and (29).

the algebra. One can easily check these conditions by reversing the order of three operators in different ways (see (14)) and by reversing the ordering of two operators twice.

In this way, we see that the following extension of the ZFA with r-matrices given in (5) is self-consistent. Extend equation (3) to

$$r_{ij}^{i'j'}(z_1 - z_2 - \tfrac{1}{2}nw)B^j(z_2)A_{i'}(z_1) = \lambda(z_1 - z_2)A_i(z_1)B^{j'}(z_2) + \delta_{ij'}\delta(z_1 - z_2 - \tfrac{1}{2}nw)\rho \tag{32}$$

with a compatible change of equation (4), where λ is an arbitrary c number function of z and ρ is a constant. Let r_1 and $-r_2$ be $r(z)$ matrices with a different modular parameter τ, and $r_3 = \mathrm{id}$, we clearly have a self-consistent ZFA. We then put $\lambda(z_1 - z_2)$ as the function $\tau(\xi_1/\xi_2)$ given in [12]. The ZFA becomes the same as equations (18)–(20) given by Foda *et al* [12]:

$$\sum R_{\varepsilon_1\varepsilon_2,\varepsilon_1'\varepsilon_2'}\left(\frac{\xi_1}{\xi_2}\right)\Phi_{\varepsilon_1'}(\xi_1)\Phi_{\varepsilon_2'}(\xi_2) = \Phi_{\varepsilon_2}(\xi_2)\Phi_{\varepsilon_1}(\xi_1)$$

$$\begin{aligned}\tau\left(\frac{\xi_1}{\xi_2}\right)\Psi_{\varepsilon_2}^*(\xi_2)\Phi_{\varepsilon_1}(\xi_1) &= \Phi_{\varepsilon_1}(\xi_1)\Psi_{\varepsilon_2}^*(\xi_2) - \sum \Psi_{\varepsilon_2}^*(\xi_2)\Psi_{\varepsilon_1}^*(\xi_1)R^*_{\varepsilon_1\varepsilon_2,\varepsilon_1'\varepsilon_2'}\left(\frac{\xi_1}{\xi_2}\right)\\ &= \Psi_{\varepsilon_1'}^*(\xi_1)\Psi_{\varepsilon_2'}^*(\xi_2).\end{aligned}$$

One can also multiply Ψ^* and Φ to obtain the L operators satisfying

$$R_{12}^+\left(\frac{\xi_1}{\xi_2}\right)L^1(\xi_1)L^2(\xi_2) = L^2(\xi_2)L^1(\xi_1)R_{12}^{*+}\left(\frac{\xi_1}{\xi_2}\right)$$

provided

$$L(\xi)_i^j = \Phi_i(\xi)\Psi_j^*(\xi q^{-1/2}). \tag{33}$$

The intertwining relations

$$\Phi_j(\xi_2)L_{ii''}(\xi_1) = \sum R_{ij;i'j'}^+\left(\frac{\xi_1}{\xi_2}\right)L_{i'i''}(\xi_1)\Phi_{j'}(\xi_2)$$

$$L_{j,j''}(\xi_2)\Psi_{i''}^*(\xi_1) = \sum \Psi_{i'}^*(\xi_1)L_{j,j'}(\xi_2)R_{j'i';j''i''}^{*+}\left(q^{-\frac{1}{2}}\frac{\xi_2}{\xi_1}\right)$$

can also be obtained. These are similar to those in [12]. Thus these equations are self-consistent.

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